Analysis 3 Review

notes taken by Christopher Hundt

from lectures by K. GowriSankaran
MATH 354, McGill University, Fall 2004

Updated December 4, 2004

LEMMA (HÖLDER'S INEQUALITY):
Let $a, b > 0$;
$p, q > 1$ be conjugate indices (i.e., $\frac{1}{p} + \frac{1}{q} = 1$).
Then $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$.

THEOREM (MINKOWSKI FOR INTEGRALS):
Let $f, g : [0,1] \to \mathbb{R}$ be continuous.
Then $\left(\int_0^1 |f+g|^p\,dx\right)^{1/p} \leq \left(\int_0^1 |f|^p\,dx\right)^{1/p} + \left(\int_0^1 |g|^p\,dx\right)^{1/p}$.

THEOREM (MINKOWSKI FOR SUMS):
Let $(a_j), (b_j) \in l_p$.
Then $(a_j + b_j) \in l_p$ and $\left(\sum_{j=1}^{\infty} |a_j + b_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} |b_j|^p\right)^{1/p}$.

DEFINITION: Fix a metric space (X, d) and a subset $A \subseteq X$. Then we say $z \in X$ is a **limit point** of A if there is a distinct sequence $x_n \in A$ such that $x_n \to z$.

DEFINITION: Let $A \subseteq X$. We call the **closure** of A in X, $\overline{A}$, $\overline{A} = A \cup \{z : z \text{ is a limit point of } A \text{ in } X.\}$

THEOREM: Let $A_1, A_2 \subseteq X$.
Then $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

REMARK: Let $A_1, A_2 \subseteq X$.
Then $\overline{A_1 \cap A_2} \subseteq \overline{A_1} \cap \overline{A_2}$.

Theorem: Let $y \in X$;
$A \subseteq X$.
Then y is a limit point of A iff $\forall \varepsilon > 0$ $B(y, \varepsilon) \cap A$ contains an element $\neq y$.

Corollary: Let $A \subseteq X$;
$y \notin A$.
Then $y \in \overline{A} \iff \forall \varepsilon > 0\ \exists x(\neq y) \in B(y, \varepsilon) \cap A$.

Definition: $A \subseteq X$ is **closed** in X if the closure of A in X is A.

Theorem: Let F be the collection of all closed subsets (relative to X).
Then

1. $X, \emptyset \in F$;
2. $A_1, \ldots, A_n \in F \implies \bigcup_{j=1}^{n} A_j \in F$; and
3. $A_i \in F\ \forall i \in I \implies \bigcap_{i \in I} A_i \in F$.

Definition: If $X \setminus U$ is closed then U is **open**.

Lemma: Let $V \subseteq X$.
Then V is open iff $\forall x \in V\ \exists \varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq V$.

Lemma: Let $A \subseteq X$.
Then $\overline{\overline{A}} = \overline{A}$.

Theorem: Let X_n be a countable set of countable sets.
Then $\bigcup_n X_n$ is countable.

Definition: We say that two sets X and Y have the same "size" if $\exists f : X \to Y$ such that f is 1-1 and onto. In this case we say X and Y have the same **cardinality** or **cardinal number**.

Theorem: Let X be a non-void set.
Then $\text{card}(X) < \text{card}(\mathcal{P}(X))$.

Theorem: $\mathbb{R}$ is not countable.

Remark: Let $Y \subseteq X$.
Then

1. $\forall y \in Y, \varepsilon > 0$ $B_Y(y, \varepsilon) = B_X(y, \varepsilon) \cap Y$; and
2. V is open (closed) in Y iff $\exists$ U open (closed) in X such that $Y \cap U = V$.

DEFINITION: A set $A \subseteq B(\subseteq X)$ is said to be **dense** in B if $B \subseteq \overline{A}$. A is dense in X if $\overline{A} = X$.

DEFINITION: (X, d) is said to be **separable** if there is a countable dense subset of X.

DEFINITION: (X, d) is said to have a **countable base** (for open sets) if there exist $V_1, V_2, \ldots$ open sets such that every open set of X is a union of V_n's.

THEOREM: Let (X, d) be a metric space.
Then (X, d) is separable iff (X, d) has a countable base for open sets.

DEFINITION: $f : (X, d) \to (Y, \rho)$ is **continuous** at $x_0 \in X$ if $\forall$ sequence $(z_n) \in X$ we have $z_n \stackrel{d}{\to} x_0 \implies f(z_n) \stackrel{\rho}{\to} f(x_0)$. f is continuous globally if f is continuous at every point in X.

THEOREM: $f : (X, d) \to (Y, \rho)$ is continuous at x_0 iff $\forall \varepsilon > 0\ \exists \delta > 0$ such that $x \in B(x_0, \delta) \implies f(x) \in B(f(x_0), \varepsilon)$.

THEOREM: The following are equivalent (for $f : X \to Y$):

1. f is continuous.
2. $S \subseteq Y$ is open $\implies f^{-1}(S)$ is open.
3. $S \subseteq Y$ is closed $\implies f^{-1}(S)$ is closed.

DEFINITION: A sequence $(x_n) \in X$ is a **Cauchy sequence** (or a fundamental sequence) if $\forall \varepsilon > 0\ \exists N$ such that $d(x_m, x_n) < \varepsilon$ for $m, n > N$.

DEFINITION: A metric space is **complete** if every Cauchy sequence converges.

THEOREM:

1. A convergent sequence is a Cauchy sequence.
2. If (x_n) is Cauchy and it has a convergent subsequence then (x_n) converges.

THEOREM: Let (X, d) be a complete metric space;
$Y \subseteq X$.
Then (Y, d) is complete iff Y is closed in X.

THEOREM: (X, d) is complete iff every sequence of nested closed spheres with radii $\to 0$ has a nonvoid intersection.

DEFINITION: Let $(X,d),(X',d')$ be metric spaces. A 1-1 and onto mapping $f : X \to X'$ is said to be **isometric** if $d(x,y) = d'(f(x), f(y))$ for all $x, y \in X$.

DEFINITION: Let X, X^* be metric spaces, where X^* is complete. Then X^* is said to be a **completion** of X if $X \subseteq X^*$ and X is dense in X^* (with these statements holding perhaps only under some isometry).

THEOREM: Every metric space (X,d) has a unique (up to isometry) completion.

DEFINITION: A mapping A from a metric space (X,d) onto itself is a **contraction** if $d(Ax, Ay) \leq \alpha d(x,y)$ for all $x, y \in X$, where $\alpha < 1$. It's always continuous.

THEOREM (PRINCIPLE OF CONTRACTION MAPPING): Every contraction mapping on a complete metric space has a unique fixed point.

DEFINITION: $f : (X,d) \to (Y,\rho)$ is **uniformly continuous** on X if $\forall \varepsilon > 0\ \exists \delta > 0$ (independent of points in X) such that $x, x' \in X$ and $d(x,x') < \delta \implies \rho(f(x), f(x')) < \varepsilon$.

DEFINITION: Let (X,d) be a metric space. A set $K \subseteq X$ is called a **compact set** if $\forall$ sequence $(x_n) \in K$ there is a subsequence $(x_{n_k}) \in K$ that converges to some element $y \in K$.

PROPOSITION: Let K be a compact subset of X.
Then K is closed in X.

PROPOSITION: Let K be a compact subset of X;
$A \subseteq K$.
Then A is closed $\iff$ A is compact.

THEOREM: Let $f : (X,d) \to (Y,\rho)$ be a continuous map;
$K \subseteq X$ be compact.
Then $f(K)$ is a compact subset of Y.

THEOREM: Let $K \subseteq \mathbb{R}$.
Then K is compact $\iff$ K is bounded and closed.

REMARK: The cantor set $C \subseteq [0,1]$ is compact since it is closed and bounded.

THEOREM: Let $(X_1, d_1), (X_2, d_2)$ be compact.
Then $X_1 \times X_2$ is compact with metrics $\sqrt{d_1^2 + d_2^2}$, $\max(d_1, d_2)$ or $d_1 + d_2$.

THEOREM: Let (X_j, d_j) be compact metric spaces with $d_j \leq 1$;
$X = \prod_{j=1}^{\infty} X_j$ with $d = \sum_{j=1}^{\infty} \frac{1}{2^j} d_j$.
Then (X, d) is compact.

THEOREM: Let (X, d) be a compact metric space.
Then (X, d) is complete.

DEFINITION: Let (X, d) be a metric space and $\alpha > 0$. We say that the subset $A \subseteq X$ is an **α-net** for $B \subseteq X$ if $\forall b \in B\ \exists a \in A$ such that $d(a, b) < \alpha$.

DEFINITION: A subset $K \subseteq (X, d)$ is said to be **totally bounded** if $\forall \varepsilon > 0\ \exists$ a finite set $\{a_1, \ldots, a_{n_\varepsilon}\} \subseteq X$ which is an ε-net for K.

THEOREM: Let (X, d) be a metric space;
$K \subseteq X$ be compact.
Then K is totally bounded (and (K, d) is complete).

THEOREM: Let $A \subseteq (X, d)$ be totally bounded and d-complete.
Then A is compact.

THEOREM: Let (X, d) be a compact metric space.
Then X is separable.

DEFINITION: Let (X, d) be a metric space and $A \subseteq X$. We say that a collection $\{V_\alpha\}_{\alpha \in I}$ (where I is an arbitrary indexing set, usually uncountable) with V_α open $\forall \alpha$ is an **open cover** for A if $A \subseteq \bigcup_{\alpha \in I} V_\alpha$.

THEOREM (LINDELÖFF):
Let (X, d) be a separable metric space;
$\{V_\alpha\}_{\alpha \in I}$ be an open cover of X with I uncountable.
Then $\exists$ a countable set $I_1 \subset I$ such that $\cup_{\alpha \in I_1} V_\alpha = X$. That is, there is a countable subcover.

THEOREM: Suppose (X, d) is such that for every open cover of X there exists a finite subcover. Then (X, d) is compact.

THEOREM: Let (X, d) be a compact space;
$\{V_\alpha\}_{\alpha \in I}$ be an open cover of X.
Then $\exists$ a finite subcover.

Theorem (Lebesgue lemma):
Let (X,d) be compact;
$\{V_\alpha\}_{\alpha \in I}$ an open cover of X with I uncountable.
Then $\exists\ \delta > 0$ such that $d(x,x') < \delta \implies \exists\ V_\alpha$ for some $\alpha \in I$ such that $x, x' \in V_\alpha$.

Theorem: Let (X,d) be compact;
$f : (X,d) \to (Y,d')$ be continuous.
Then f is uniformly continuous.

Theorem: Let (X,d) be a metric space.
Then the following are equivalent:

1. From every open cover of X we can get a finite subcover. (i.e., X is compact.)
2. If F_i is closed for all $i \in I$ such that $\forall$ finite subset $J \subseteq I$ $\bigcap_{i\in J} F_i \neq \emptyset$ then $\bigcap_{i \in I} F_i \neq \emptyset$.

Theorem: A subset $K \subseteq \mathbb{R}^n$ is compact iff K is bounded and closed.

Theorem: Let (X,d) be a compact set;
$f : X \to \mathbb{R}$ be continuous.
Then f is bounded and attains its bound.

Definition: Let (X,d) and (Y,d') be compact metric spaces. Then C_{XY} is definied as the set of all continuous mappings $X \to Y$. This is a metric space with distance function $\rho(f,g) = \sup\{d'(f(x),g(x)) : x \in X\}$.

Definition: Let $(X,d),(Y,d')$ be metric spaces and $A \subseteq C_{XY}$. Then A is **equi-continuous** at $x_0 \in X$ if $\forall \varepsilon > 0\ \exists \delta > 0$ (independent of $f \in A$) such that $\forall y \in B(x_0,\delta)\ \forall f \in A\ d'(f(x_0), f(y)) < \varepsilon$.

Definition: Let $(X,d),(Y,d')$ be metric spaces and $A \subseteq C_{XY}$. A is **uniformly equi-continuous** if $\forall \varepsilon > 0\ \exists \delta > 0$ (independent of $f \in A$ and $x \in X$) such that $\forall x' \in B(x,\delta)\ \forall f \in A\ d'(f(x), f(x')) < \varepsilon$.

Remark: Any finite set of uniformly continuous functions is uniformly equi-continuous.

Theorem: Let (X,d) be compact;
$A \subseteq C_{XY}$ be equi-continuous at each $x \in X$.
Then A is uniformly equi-continuous on X.

Remark:

1. C_{XY} is closed in M_{XY}, the set of *all* mappings $X \to Y$.
2. Any $f \in C_{XY}$ is uniformly continuous.

Theorem (Arzelà-Ascoli):
Let X, Y be compact metric spaces;
$D \subseteq C_{XY}$.
Then D is totally bounded in C_{XY} iff D is (uniformly) equi-continuous.

Definition: Let (X, d) be a metric space. $A \subseteq X$ is said to be **connected** if $\nexists$ two open sets $V, W \subseteq X$ such that $V \cap W \cap A = \emptyset$, $A \subseteq V \cup W$, $V \cap A \neq \emptyset$, and $W \cap A \neq \emptyset$. In the case where $A = X$ we say that X is a **connected space**.

Remark: X is a connected space iff $\nexists$ a non-trivial clopen (closed and open) subset of X.

Theorem: Let $A \subseteq (X, d)$ be connected;
$B \subseteq X$ such that $A \subseteq B \subseteq \overline{A}$.
Then B is connected.

Theorem: Let $\{A_i\}_{i \in I}$ be connected subsets of X such that $\bigcap_{i \in I} A_i \neq \emptyset$.
Then $\bigcup_{i \in I} A_i$ is connected.

Corollary: Let X be a metric space;
$x_0 \in X$;
M_{x_0} be the union of all connected subsets of X containing x_0.
Then M_{x_0} is the "largest" connected set of X containing x_0.

Definition: The largest connected set of X containing x_0 is called the **connected component** of X containing x_0.

Remark:

1. Any two connected components are disjoint or identical.
2. If A is a connected subset of X then $\exists$ a maximal connected set of X containing A.

Theorem: Let $(X, d), (Y, d')$ be metric spaces;
$f : X \to Y$ be continuous;
$A \subseteq X$ be connected.
Then $f(A)$ is connected.

Theorem: Let $A \subseteq \mathbb{R}$.
Then A is connected $\iff$ A is an interval.

Corollary: Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous;
$I \subseteq A$ be a closed and bounded interval.
Then $f(I)$ is bounded and attains all values between the upper and lower bounds, as well as the bounds themselves.

Definition: A metric space X is said to be **locally connected** if $\forall x \in A \; \forall$ ball $B(x, r)$ with $r > 0 \; \exists$ a connected open set V containing x such that $V \subseteq B(x, r)$.

Remark: In any space X, given any connected set A, there exists a unique maximal connected set containing A.

Theorem: Let X be a locally connected metric space.
Then every connected component is open.

Definition: A **path** in a metric space X is a continuous function $\gamma : [0, 1] \to X$.

Definition: An **arc** in X is the image of a path, i.e., $\{\gamma(t) : t \in [0, 1]\}$.

Remark: A path is one parameterization of an arc.

Definition: We say that a space is **arcwise connected** (or pointwise connected) if $\forall x, y \; \exists$ a path with all values in X, $\gamma : [0, 1] \to X$, with $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem: An arcwise connected space is connected.

Remark: Every convex set $C \subseteq \mathbb{R}^n$ is arcwise connected.

Remark: $\mathbb{R}^n$ is connected $\forall n$.

Theorem: A nonvoid open set $V \subseteq \mathbb{R}^n$ is connected $\iff$ it is arcwise connected.

Definition: Let X, Y be vector spaces in $\mathbb{R}^d$. Then $L : X \to Y$ is **linear** if $\forall \overrightarrow{x}, \overrightarrow{x_1}, \overrightarrow{x_2} \in \mathbb{X}$ and $\forall c \in \mathbb{R}$ we have $L(\overrightarrow{x_1} + \overrightarrow{x_2}) = L(\overrightarrow{x_1}) + L(\overrightarrow{x_2})$ and $L(c\overrightarrow{x}) = cL(\overrightarrow{x})$.

Remark: If $L : X \to Y$ is linear, 1-1, and onto then $\Phi : Y \to X = L^{-1}$ is linear.

Theorem: Let $L : X \to X$ be linear for a vector space X.
Then L is 1-1 $\iff$ L is onto.

Definition: Let $X, Y \subseteq \mathbb{R}^d$ be two vector spaces. Then $L(X, Y)$ is the vector space formed by the set of all linear mappings $X \to Y$.

Definition: For any $L \in L(\mathbb{R}^n, \mathbb{R}^m)$ we define the **norm** of L as $\|L\| = \sup\limits_{|\overrightarrow{x}| \leq 1} |L(\overrightarrow{x})|$.

Theorem:

1. $\|L\| < \infty \ \forall L \in L(X, Y)$.
2. $\|L\| = \sup_{|\overrightarrow{x}|=1} |L(\overrightarrow{x})|$.
3. $\|L\| = \inf\{\lambda : |L(\overrightarrow{x})| \leq \lambda |\overrightarrow{x}| \ \forall \overrightarrow{x} \in X\}$.
4. L is a uniformly continuous function on X for all $L \in L(X, Y)$.

Remark: (From the proof of the preceding theorem) $|L(\overrightarrow{x})| \leq \|L\| |\overrightarrow{x}|$ for any L and $\overrightarrow{x}$.

Proposition: Let $L, L_1, L_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$.
Then

1. $\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$.
2. $\|cL\| = |c| \|L\|$.
3. $(L_1, L_2) \to \|L_1 - L_2\|$ defines a metric on $L(\mathbb{R}^n, \mathbb{R}^m)$.
4. If $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ then the composite function $ML \in L(\mathbb{R}^n, \mathbb{R}^p), ML(\overrightarrow{x}) = M(L(\overrightarrow{x}))$, satisfies $\|ML\| \leq \|M\| \|L\|$.

Definition: $\Omega \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ is the set of all invertible elements in $L(\mathbb{R}^n, \mathbb{R}^n)$.

Theorem: Let $L \in \Omega$;
$\|L^{-1}\| = \frac{1}{\alpha}$;
$M \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|M - L\| = B < \alpha$.
Then M^{-1} exists, i.e., $M \in \Omega$ (so Ω is open in $L(\mathbb{R}^n, \mathbb{R}^n)$), and $L \mapsto L^{-1}$ is a continuous homomorphism.

Remark: If (a_{ij}) is the matrix representation of a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, then $\|L\| \leq \big(\sum\limits_{i,j} a_{ij}^2\big)^{1/2}$.

THEOREM: Let X be a metric space;

$a_{ij} : X \to \mathbb{R}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Then $x \mapsto (a_{ij}(x))$ is a function from X to $L(\mathbb{R}^n, \mathbb{R}^m)$ with respect to standard bases. Then if the a_{ij}'s are continuous then the mapping $x \mapsto (a_{ij}(x))$ is continuous with respect to the norm.

DEFINITION: Let $\overrightarrow{f} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where V is open and let $\overrightarrow{x} \in V$. Then if there is a linear map $A(\overrightarrow{x})$ such that

$$\frac{|\overrightarrow{f}(\overrightarrow{x} + \overrightarrow{h}) - \overrightarrow{f}(\overrightarrow{x}) - A\overrightarrow{h}|}{|\overrightarrow{h}|} \to 0$$

as $|\overrightarrow{h}| \to 0$ then we say that $\overrightarrow{f}$ is **differentiable** at x and $A(\overrightarrow{x})$ is the **derivative** of $\overrightarrow{f}$ at $\overrightarrow{x}$.

REMARK: We can write $\overrightarrow{f}(\overrightarrow{x} + \overrightarrow{h}) - \overrightarrow{f}(\overrightarrow{x}) - A\overrightarrow{h} = \overrightarrow{r}(\overrightarrow{h})$, where $\overrightarrow{r}(\overrightarrow{h})$ is the "error" term, and the existence of the derivative implies that $|\overrightarrow{r}(\overrightarrow{h})|$ is small compared to $|\overrightarrow{h}|$.

REMARK: $\overrightarrow{f}$ is continuous at $\overrightarrow{x}$ if it is differentiable there.

THEOREM: Let $\overrightarrow{f}$ be differentiable at $\overrightarrow{x} \in E$, where E is open;

A_1, A_2 be two maps satisfying the definition of the derivative..

Then $A_1 = A_2$, i.e., the derivative is unique.

THEOREM (CHAIN RULE):

Let $\overrightarrow{g} : V \subseteq \mathbb{R}^m \to \mathbb{R}^p$ be differentiable, with V open;

$\overrightarrow{f} : W \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, with W open;

$\overrightarrow{F} : W \to \mathbb{R}^p$ be defined by $\overrightarrow{F}(\overrightarrow{x}) = \overrightarrow{g}\big(\overrightarrow{f}(\overrightarrow{x})\big) \in \mathbb{R}^p$.

Then $\overrightarrow{F}$ is differentiable and $\overrightarrow{F}'(x) = \overrightarrow{g}'\big(\overrightarrow{f}(x)\big) \cdot \overrightarrow{f}'(x)$.

DEFINITION: A function $\overrightarrow{f} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where V is open, is **continuously differentiable** (C^1) if it is differentiable and $\overrightarrow{x} \mapsto \overrightarrow{f}'(\overrightarrow{x})$, a mapping from V to $L(\mathbb{R}^n, \mathbb{R}^m)$, is a continuous map.

THEOREM: Let $\overrightarrow{f} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where V is open.

Then $\overrightarrow{f}$ is $C^1 \iff$ for $\overrightarrow{f} = (f_1, \ldots, f_m)$ and $\forall i, j$ $\frac{\partial f_i}{\partial x_j}$ exist and are continuous on V.

Theorem (inverse function theorem):
Let $\vec{f} : V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be C^1, where V is open;
$\vec{a}$ be in the domain of the definition of $\vec{f}$;
$\vec{f}\,'(\vec{a}) \in L(\mathbb{R}^n, \mathbb{R}^n)$ be invertible.

Then $\exists$ open sets U containing $\vec{a}$ and V containing $\vec{f}(\vec{a})$ such that $\vec{f} : U \to V$ is a bijection.

Theorem (implicit function theorem):
Let $f : E \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$ where E is open, such that $\vec{f} \in C^1(E)$;
$(\vec{a}, \vec{b})$ be such that $\vec{f}(\vec{a}, \vec{b}) = \vec{0}$;
$A = \vec{f}\,'(\vec{a}, \vec{b})$ (an $n \times (n+m)$ matrix);
A be such that $A(\vec{h}, \vec{0}) = \vec{0} \iff \vec{h} = \vec{0}$.

Then $\exists$ an open set W containing $\vec{b}$ and a C^1 map $\vec{g} : W \to \mathbb{R}^n$ such that $\vec{f}\left(\vec{g}(\vec{y}), \vec{y}\right) = \vec{0}\ \forall \vec{y} \in W$ and $\vec{g}(\vec{b}) = \vec{a}$.

Definition: $\mathcal{C}(X)$ is the set of all real-valued continuous functions on X and is a metric space with metric $\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|$.

Definition: $A \subseteq \mathcal{C}(X)$ is an **algebra** (of continuous functions) if $\forall f, g \in A\ \forall c \in \mathbb{R}$ we have $f+g, fg, cf \in A$ (where $fg(x) = f(x)g(x)$). Note that $\mathcal{C}(X)$ itself is an algebra.

Lemma: If $A \subseteq \mathcal{C}(X)$ is an algebra then so is $\overline{A} \subseteq \mathcal{C}(X)$.

Theorem (Stone-Weierstrass v. 1): Let X be a compact metric space and $A \subseteq \mathcal{C}(X)$ be an algebra such that

1. A “separates points” of X, i.e., given $x_1 \neq x_2$ in X $\exists f \in A$ such that $f(x_1) \neq f(x_2)$; and
2. A “vanishes nowhere,” i.e., $\forall x\ \exists g \in A$ such that $g(x) \neq 0$.

Then $\overline{A} = \mathcal{C}(X)$ (i.e., A is uniformly dense in $\mathcal{C}(X)$).

Theorem (Stone-Weierstrass v. 2): Let X be a compact metric space and $A \subseteq \mathcal{C}(X)$ such that

1. A is a vector space;
2. A separates points;
3. $1 \in A$;
4. A is a lattice for the natural order (i.e., $f, g \in A \implies \max(f,g), \min(f,g) \in A$); and
5. A is closed.

Then $\overline{A} = \mathcal{C}(X)$ (i.e., A is uniformly dense in $\mathcal{C}(X)$).

Index