COMP-362: Honours Algorithm Design

Assignment 3: Circulations on digraphs

Winter Term 2004

Let $G = (V, E)$ be a directed graph. We assume that each pair $(u, v) \in V \times V$ is assigned a *cost* $c(u, v)$ and moreover the cost function is anti-symmetric, i.e. $c(u, v) = -c(v, u)$. Also, each pair $(u, v) \in V \times V$ has a *capacity* $u(u, v) \geq 0$, and we set $u(u, v) = 0$ if $(u, v) \notin E$.

The notion of a circulation on G is very similar to that of a flow. More specifically, a *circulation* on G is a function $f : V \times V \to \mathbb{R}$ such that:

- for every $(u, v) \in V \times V$ we have $f(u, v) \leq u(u, v)$ (capacity constraint),
- for every $(u, v) \in V \times V$ we have $f(u, v) = -f(v, u)$ (f is anti-symmetric),
- for every vertex $v \in V$ we have $\sum_{u \in V} f(v, u) = 0$ (this is the conservation law).

Note that this is like the ordinary flow except that G has neither a source vertex nor a sink vertex. The *cost* of a circulation f is defined as

$$c(f) = \sum_{(u,v) \in V \times V} c(u, v) f(u, v).$$

The problem is the following: Given a directed graph G as above find a circulation of minimum cost.

Given a circulation f on G we can define the *residual graph* $G_f = (V, E_f)$ much as we defined the residual network in the case of flows: we set $E_f = \{(u, v) \in V \times V \;:\; f(u, v) < u(u, v)\}$. The *residual capacity* of the edge $(u, v) \in E_f$ is $u_f(u, v) = u(u, v) - f(u, v)$. Note that in G_f every residual capacity is positive.

Now, we introduce a notion that we have seen in Johnson's algorithm. A *potential* is just a real-valued function on V. If $p : V \to \mathbb{R}$ is a potential, then the *reduced cost* c_p of the pair $(u, v) \in V \times V$ is defined as $c_p(u, v) = c(u, v) + p(u) - p(v)$.

i) Prove that along a directed cycle C of G the cost of the cycle $c(C) = \sum_{(u,v) \in C} c(u, v)$ equals the reduced cost of C, which we define as $c_p(C) = \sum_{(u,v) \in C} c_p(u, v)$.

ii) Prove that for any circulation f and any potential p we have $c(f) = c_p(f)$.

In the following questions we shall try to characterise minimum-cost circulations.

iii) If f is a minimum cost circulation, then there are no negative cost cycles in the residual graph G_f (a negative cost cycle in G_f is a cycle C of G_f with $c(C) = \sum_{(u,v) \in C} c(u, v) < 0$). We will try to prove that if there is a negative cost cycle, then we could decrease the cost of the circulation f which is assumed to be minimum. In particular, pick such a cycle C and let $\delta = \min_{e \in C} u_f(e)$. Then define the circulation $\tilde{f}$ on G as follows:

$$\tilde{f}(u,v) = \begin{cases} f(u,v) + \delta & \text{if } (u,v) \in C \\ f(u,v) - \delta & \text{if } (v,u) \in C \\ f(u,v) & \text{otherwise} \end{cases} .$$

Verify that $\tilde{f}$ is a circulation on G. Show that $c(\tilde{f}) < c(f)$, thus deriving a contradiction.

iv) Show that if there are no negative cost cycles in G_f, then there exists a potential $p : V \to \mathbb{R}$ such that $c_p(u,v) \geq 0$ for every $(u,v) \in E_f$. We shall do something similar to what we did in Johnson's algorithm: we create a graph $G' = (V \cup \{s\}, E_f \cup E')$, where $E' = \{(s,v) \ : \ v \in V\}$ and the cost of each pair (s,v), (v,s), with $v \in V$, is assumed to be zero. The potential function that has the required property can be obtained as follows: for every edge $(u,v) \in E_f \cup E'$ we regard the cost $c(u,v)$ as the weight of this edge. Then for every vertex $v \in V$ the function $p(v)$ is the weight of a shortest path (i.e., a minimum weight path) from s to v, if G' is regarded as a weighted graph with weights defined as above. Show that $c_p(u,v) \geq 0$ for every $(u,v) \in E_f$ (use the argument that we used in Johnson's algorithm; note that we assume that G_f has no negative cost cycles, i.e., there are no negative weight cycles in G' under the above weighting.)

v) Finally, we prove that if f is a circulation and there exists a potential p such that $c_p(u,v) \geq 0$ for every $(u,v) \in E_f$, then f has to be a minimum cost circulation.

The proof of this is by contradiction. Assume that f' is a circulation on G such that $c(f') < c(f)$ and consider $f'' = f' - f$, i.e. for every $(u,v) \in V \times V$ we set $f''(u,v) = f'(u,v) - f(u,v)$. Show that $f''(u,v) \leq u_f(u,v) \equiv u(u,v) - f(u,v)$ for any $(u,v) \in V \times V$. Thus, if $f''(u,v) > 0$, then $(u,v) \in E_f$. Use these observations and the result of question ii), with p being the potential of our assumption, to show that $c(f'') \geq 0$. This contradicts the assumption that $c(f'') < 0$ and the statement we want to prove is now clear. Thus we have shown the following theorem:

Theorem 1 *For a circulation f on G the following are equivalent:*

1. *f is of minimum cost,*
2. *there are no negative cost cycles in G_f,*
3. *the exists a potential function $p : V \to \mathbb{R}$ such that $c_p(u,v) \geq 0$, for every $(u,v) \in E_f$.*

Now, describe an algorithm that finds a minimum cost circulation on a digraph G with costs and capacities assigned to its edges. Do not give details, only a generic description similar to the one we used for the Ford-Fulkerson algorithm in class. (If there is a question that you cannot do, just assume that it is true and continue with the rest.)

Due date: 16th of March 2004.