

COMP 362 Assignment 2

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1. Assume $G = (V, E)$. Call $\text{CDFS}(G)$ to run.

$\text{CDFS}(G)$

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1  for each  $v \in V$ 
2      do  $\text{colour}[v] \leftarrow \text{WHITE}$ 
3   $\text{time} \leftarrow 0$ 
4  for each  $v \in V$ 
5      do if  $\text{colour}[v] = \text{WHITE}$ 
6          then if  $\text{CDFS-VISIT}(v) = \text{FALSE}$ 
7              then return "Directed Cycle!"
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$\text{CDFS-VISIT}(v)$

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1   $\text{colour}[v] \leftarrow \text{GRAY}$ 
2  for each  $u$  such that  $(v, u) \in E$ 
3      do if  $\text{colour}[u] = \text{WHITE}$ 
4          then  $\text{CDFS-VISIT}(u)$ 
5      else if  $\text{colour}[u] = \text{GRAY}$ 
6          then return FALSE
7   $\text{colour}[v] \leftarrow \text{BLACK}$ 
8  return TRUE
```

Suppose there is a directed cycle in G . Then there is some $u, v \in V$ such that $u \rightsquigarrow v \rightsquigarrow u$. Suppose u is the first one of the two visited by CDFS . Then u is coloured gray and v is still white. CDFS-VISIT does not finish with u until it has visited every vertex that u is adjacent to, and hence every vertex that those vertices are adjacent to, etc. Thus CDFS-VISIT reaches v before finishing u , so u is coloured gray when $\text{CDFS-VISIT}(v)$ starts. Then, because $v \rightsquigarrow u$, there is some w such that $v \rightsquigarrow w$ and $(w, u) \in E$. $\text{CDFS-VISIT}(w)$ will be called before finishing with v , and then, on line 5, it will find that u is coloured gray and the cycle will be detected.

Now suppose that CDFS reports a cycle. Then CDFS-VISIT found a gray node u such that, for the current node v , $(v, u) \in E$. But CDFS-VISIT colours each node black when it finishes with it, so this means that $\text{CDFS-VISIT}(v)$ was a recursive call, and $\text{CDFS-VISIT}(u)$ has not finished yet. But since $\text{CDFS-VISIT}(u)$ just follows every path from u , this means there is a path from u to v and an edge from v to u . Thus there is a cycle in the graph.

Thus, CDFS reports a cycle if and only if one exists.

2. (a) The uv -entry of $A^k(G)$ is the number of paths of length k (that is, k edges) from u to v . We can prove this by induction. For $k = 1$, $a_{uv} = 1$ if there is an edge from u to v , and 0 otherwise. But having an edge from u to v is precisely the only way there can be a path of length 1 from u to v . So the property holds for $k = 1$. So we assume, by induction that the property holds for $k = j$. Now, we consider a_{uv}^{j+1} , noting that $A^{j+1} = A^j A$.

$$a_{uv}^{j+1} = \sum_{w \in V} a_{uw}^j a_{wv} \quad (1)$$

$$= \sum_{w \in V} (\# \text{ paths of length } j \text{ from } u \text{ to } w) \times (1 \text{ if } (w, v) \in E, 0 \text{ otherwise}) \quad (2)$$

$$= |\{(u, v_1, v_2, \dots, v_{j-1}, w, v) \mid (u, v_1), (v_{j-1}, w), (w, v) \in E \text{ and } (v_i, v_{i+1}) \in E \text{ for all } i < j - 1\}|$$

$$= \text{number of paths of length } j + 1 \text{ from } u \text{ to } v.$$

Note that equation (1) comes from the definition of matrix multiplication, and equation (2) follows from the inductive hypothesis.

- (b) Now, the following algorithm will find the length of the smallest cycle of G and the number of cycles of that length, given $n = |V|$.

CYCLES(A)

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1   $B \leftarrow I$ , the identity matrix of size  $n$ 
2   $k \leftarrow 0$ 
3  while  $B \neq 0$ 
4      do  $k \leftarrow k + 1$ 
5           $B \leftarrow BA$ 
6          for  $i \leftarrow 1$  to  $n$ 
7              do  $m \leftarrow m + a_{ii}/k$ 
8              if  $m > 0$ 
9                  then return " $m$  cycles of length  $k$ "
10 return "no cycles"
```

Suppose there are x cycles of length y , and y is the smallest length of any cycle. Then each of the x cycles will involve y vertices, so there will be xy ways for a vertex to follow a path of length y and end up back at itself. Thus, the total along the diagonal column of $A^y(G)$ will be xy . CYCLES will not return before $k = y$ because the diagonal entries will remain 0 for $A^j(G)$ where $j < y$, because y is the smallest length of any cycle. Once CYCLES gets to $k = y$, it will total the diagonal entries and get $m = xy/k = xy/y = x$, and return " x cycles of length y ," the correct answer.

Now suppose there are no cycles in the graph. Then the algorithm will never find that any diagonal entries are greater than 0, so $m = 0$ always. However, with no cycles, there can be no paths of length $> |V|$. Thus, eventually $A^k(G) = 0$ for $k > |V|$, so the while loop will end, and the algorithm will terminate with the correct answer: "no cycles."

3. We note that, if $c(v) = x$ for $v \neq s$ and v not adjacent to s , then there is some u such that $(u, v) \in E$ and $c(u) = x$ or $w(u, v) = x$. In fact, for such a u , $c(v) = \min\{c(u), w(u, v)\}$ by definition of the weight of a path. Thus, we have

$$c(v) = \max\{\min\{c(u), w(u, v)\} \mid (u, v) \in E\}. \quad (3)$$

Now let us define $c^{(l)}(v) = \max\{c^{(l-1)}(v), \max\{\min\{c^{(l-1)}(u), w(u, v)\} \mid (u, v) \in E\}\}$. We know that, if a path p from s to v of maximum weight has weight x , then $c(v) = x$. Now we claim:

Claim 1 *If the maximum weight path from s to v has length (number of edges) no more than l , then $c^{(l)}(v) = c(v)$, with $c^{(l)}(v)$ defined as above, after adding (for convenience) that $c^{(k)}(s) = \infty$ for any k and $c^{(0)}(v) = 0$ for any $v \neq s$.*

We can prove this claim inductively. For $l = 1$, we have

$$\begin{aligned} c^{(1)}(v) &= \max\left\{c^{(0)}(v), \max\left\{\min\left\{c^{(0)}(u), w(u, v)\right\} \mid (u, v) \in E\right\}\right\} \\ &= \max\left\{0, \max\left\{\min\left\{c^{(0)}(u), w(u, v)\right\} \mid (u, v) \in E\right\}\right\}. \end{aligned}$$

Now the only way we can have a maximum weight path of length 1 from s to v is if $(s, v) \in E$. Then $\min\{c^{(0)}(s), w(s, v)\} = \min\{\infty, w(s, v)\} = w(s, v)$. But, for any other $u \sim v$, we have $\min\{c^{(0)}(u), w(u, v)\} = \min\{0, w(u, v)\} = 0 \leq w(s, v)$. Thus $c^{(1)}(v) = w(s, v) = c(v)$.

Now suppose that the claim is true for $l = k$. Then consider a maximum weight path p from s to v of length $k + 1$. Let $p = (s, v_1, v_2, \dots, v_k, v)$ and let $q = (s, v_1, v_2, \dots, v_k)$. Then $c^{(k)}(v_k) \geq w(q)$ by the induction hypothesis. Because p is a maximum weight path, $w(p) = c(v)$. But $w(p) = \min\{w(q), w(v_k, v)\}$. Thus, $c^{(k+1)}(v) \geq c(v)$. But clearly $c^{(k+1)}(v) \leq c(v)$ because it only uses actual paths to v . Therefore, $c^{(k+1)}(v) = c(v)$.

Now consider the following algorithm.

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CAPACITIES( $G, s$ )
1  for each  $v \in V \setminus \{s\}$ 
2      do  $c[v] \leftarrow 0$ 
3   $c[s] \leftarrow \infty$ 
4  for  $i \leftarrow 1$  to  $|V| - 1$ 
5      do for each  $(u, v) \in E$ 
6          do if  $c[v] < \min\{c[v], w(u, v)\}$ 
7              then  $c[v] \leftarrow \min\{c[v], w(u, v)\}$ 

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Claim 2 *After l iterations of the for loop on line 4 of CAPACITIES, $c[v] = c^{(l)}(v)$.*

This is easy to see, because CAPACITIES is clearly just a direct implementation of the recursive definition of $c^{(l)}(v)$.

Thus, after $|V| - 1$ iterations of the for loop on line 4 of CAPACITIES, $c[v]$ is the maximum weight path with at most $|V| - 1$ edges, so it is the maximum weight path globally.