COMP 362 Assignment 3

Christopher Hundt 110220945

March 16, 2004

i) Suppose we have some cycle $C = (v_1, v_2, \ldots, v_n, v_1)$, with $(v_i, v_{i+1}) \in E$ and $(v_n, v_1) \in E$. Then

$$\begin{aligned} c_p(C) &= \sum_{(u,v)\in C} c_p(u,v) \\ &= \sum_{(u,v)\in C} (c(u,v) + p(u) - p(v)) \\ &= (c(v_1,v_2) + p(v_1) - p(v_2)) + (c(v_2,v_3) + p(v_2) - p(v_3)) + \ldots \\ &\quad + (c(v_{n-1},v_n) + p(v_{n-1}) - p(v_n)) + (c(v_n,v_1) + p(v_n) - p(v_1)) \\ &= c(v_1,v_2) + c(v_2,v_3) + \ldots + c(v_n,v_1) \\ &= \sum_{(u,v)\in C} c(u,v) \\ &= c(C). \end{aligned}$$

ii) We note that

$$\begin{aligned} c_p(f) &= \sum_{(u,v)\in V\times V} c_p(u,v) f(u,v) \\ &= \sum_{(u,v)\in V\times V} f(u,v)\,(c(u,v) + p(u) - p(v)) \\ &= \sum_{(u,v)\in V\times V} c(u,v) f(u,v) + \sum_{(u,v)\in V\times V} f(u,v)\,(p(u) - p(v)) \\ &= c(f) + \sum_{(u,v)\in V\times V} f(u,v)\,(p(u) - p(v)). \end{aligned}$$

We now show that the sum after $c(f)$ is equal to 0.

$$\begin{aligned} \sum_{(u,v)\in V\times V} f(u,v)\,(p(u) - p(v)) &= \sum_{u\in V}\sum_{v\in V} (p(u) - p(v)) \\ &= \sum_{u\in V}\sum_{v\in V} p(u) - \sum_{u\in V}\sum_{v\in V} p(v) \\ &= \sum_{u\in V} |V| p(u) - \sum_{v\in V} |V| p(v) \\ &= 0. \end{aligned}$$

Thus $c_p(f) = c(f) + \sum_{(u,v)\in V\times V} f(u,v)\,(p(u) - p(v)) = c(f)$.

iii) First, we note that $\tilde{f}(u,v) \leq f(u,v) + \delta \leq f(u,v) + u(u,v) - f(u,v) = u(u,v)$ for any u, v by the definition of $\tilde{f}$ and δ. Thus the capacity constraint is met. If (u,v) and (v,u) are both not in C, then $\tilde{f}(u,v) = f(u,v) = -f(v,u) = -\tilde{f}(v,u)$ by definition of $\tilde{f}$ and the fact that f is a circulation. If $(u,v) \in C$ then $\tilde{f}(u,v) = f(u,v) + \delta = -(-f(u,v) - \delta) = -(f(v,u) - \delta) = -\tilde{f}(v,u)$. Thus $\tilde{f}$ is anti-symmetric. Finally, we consider $\sum_{u \in V} \tilde{f}(v,u)$ for some v. If v is not in C then $\tilde{f}(v,u) = f(v,u)$ for all u and $\sum_{u \in V} \tilde{f}(v,u) = \sum_{u \in V} f(v,u) = 0$. Otherwise, there is some v' such that $(v', v) \in C$ and some v'' such that $(v, v'') \in C$. Then $\tilde{f}(v,v') = f(v,v') - \delta$ and $\tilde{f}(v,v'') = f(v,v'') + \delta$ and $\tilde{f}(v,u) = f(v,u)$ for all other $u \in V$. Then $\sum_{u \in V} \tilde{f}(v,u) = \sum_{u \in V} f(v,u) - \delta + \delta = \sum_{u \in V} f(v,u) = 0$. Thus the conservation condition is met and $\tilde{f}$ is a circulation.

Now we consider $c(\tilde{f})$. For the pairs (u,v) that are not connected by an edge in C the value $c(u,v)\tilde{f}(u,v) = c(u,v)f(u,v)$ remains the same because, for such vertices, $\tilde{f}(u,v) = f(u,v)$. So we need consider only those pairs (u,v) such that $(u,v) \in C$ or $(v,u) \in C$. If $(u,v) \in C$ then $\tilde{f}(u,v) = f(u,v) + \delta$. If $(v,u) \in C$, then we have $\tilde{f}(u,v) = f(u,v) - \delta$. Finally, we note that $(u,v) \in C \implies (v,u) \notin C$ (there can't be a negative cost cycle of length two because the total cost of any double edge is 0). Thus the sets $\{(u,v) \in C\}$ and $\{(u,v) : (v,u) \in C\}$ are disjoint. Thus

$$\begin{aligned}
c(\tilde{f}) &= \sum_{(u,v) \in V \times V} c(u,v)\tilde{f}(u,v) \\
&= \sum_{(u,v) \in C} c(u,v)\tilde{f}(u,v) + \sum_{(u,v) \in C} c(v,u)\tilde{f}(v,u) + \sum_{(u,v):(u,v),(v,u) \notin C} c(u,v)\tilde{f}(u,v) \\
&= \sum_{(u,v) \in C} c(u,v)\left(f(u,v) + \delta\right) + \sum_{(u,v) \in C} c(v,u)\left(f(v,u) - \delta\right) + \sum_{(u,v):(u,v),(v,u) \notin C} c(u,v)f(u,v) \\
&= \sum_{(u,v) \in V \times V} c(u,v)f(u,v) + \sum_{(u,v) \in C} c(u,v)\delta - \sum_{(u,v) \in C} c(v,u)\delta \\
&= c(f) + 2\sum_{(u,v) \in C} c(u,v)\delta \\
&< c(f),
\end{aligned}$$

since $\sum_{(u,v) \in C} c(u,v) < 0$ and $c(u,v) = -c(v,u)$. Thus $c(\tilde{f}) < c(f)$, a contradiction which means that f was not a minimum circulation.

iv) First we note that the shortest path is well-defined, because there are no negative cycles. Then, for any edge $(u,v) \in E_f$, $p(v) \leq p(u) + c(u,v)$. If not, then the path from s to u with cost $p(u)$, followed by taking the edge from u to v with cost $c(u,v)$, would be a shorter path than the one with cost $p(v)$, a contradiction. Thus, $p(v) \leq p(u) + c(u,v)$, so $0 \leq c(u,v) + p(u) - p(v) = c_p(v)$.

v) Suppose f is a circulation on G and there is a potential p such that $c_p(u,v) \geq 0$ for all $(u,v) \in E_f$. Now suppose that f' is another circulation on G such that $c(f') < c(f)$. Then we define f'', a circulation on G_f, as $f''(u,v) = f'(u,v) - f(u,v)$ for all $u, v \in V$. We see, then, that

$$c(f'') = \sum_{(u,v) \in V \times V} c(u,v)f''(u,v) = \sum_{(u,v) \in V \times V} c(u,v)f'(u,v) - \sum_{(u,v) \in V \times V} c(u,v)f(u,v) = c(f') - c(f).$$

Now we know that f'' is anti-symmetric because $f''(v,u) = f'(v,u) - f(v,u) = -f'(u,v) + f(u,v) = -f''(u,v)$. Also, f' meets the conservation law because, for any $v \in V$,

$$\sum_{u \in V} f''(u,v) = \sum_{u \in V} \left(f'(u,v) - f(u,v)\right) = \sum_{u \in V} f'(u,v) - \sum_{u \in V} f(u,v) = 0 - 0 = 0.$$

Finally, the capacity constraint is met because $f'(u,v) \le u(u,v)$ and (by definition) $u_f(u,v) = u(u,v) - f(u,v)$, so $f''(u,v) = f'(u,v) - f(u,v) \le u(u,v) - f(u,v) = u_f(u,v)$. Thus f is a circulation on G_f, so it must be that $(u,v) \in E_f$ if $f''(u,v) > 0$, because of the capacity constraint.

We now consider $c_p(f'') = \sum_{(u,v)\in V\times V} f''(u,v)c_p(u,v)$. Let us consider $u, v \in V \times V$. If $f''(u,v) = 0$, then $f''(u,v)c_p(u,v) = f''(v,u)c_p(v,u) = 0$, so the vertices do not affect the sum $c_p(f'')$, whether $(u,v),(v,u)$ are edges or not. If $f''(u,v) > 0$, then (as above) $(u,v) \in E_f$. Also $f''(u,v)c_p(u,v) \ge 0$ since $c_p(u,v) \ge 0$ by assumption. Then if $(v,u) \in E_f$ also $c_p(u,v) = c_p(v,u) = 0$ since $c_p(u,v) = -c_p(v,u)$ and both are at least 0. In that case $f''(u,v)c_p(u,v) = f''(v,u)c_p(v,u) = 0$, so this pair does not contribute to the sum $c_p(f'')$. Thus $c(f'') \ge 0$. If $f''(u,v) > 0$ but $(v,u) \notin E_f$ then the pair of vertices contributes a nonnegative amount to the sum $c_p(f'')$ since $f''(u,v)c_p(u,v) \ge 0$. Finally, if $f''(u,v) < 0$ then $f''(v,u) > 0$ and we have same case just discussed, so still we do not decrease the sum. Thus $c_p(f'') = \sum_{(u,v)\in V\times V} f''(u,v)c_p(u,v) \ge 0$, and then, by (ii), $c(f'') = c_p(f'') \ge 0$. But then $c(f'') = c(f') - c(f) \ge 0$, so $c(f') \ge c(f)$, a contradiction which tells us that f was a minimum cost circulation.

We now know that f is of minimum cost if and only if there are no negative cost cycles in G_f. So, given any circulation f with negative cost cycles, it is not minimum and can be "improved" with the new circulation $\tilde{f}$ described in (iii). However, we note that every time we make this improvement we give one element of the cycle circulation equal to its capacity (because $\delta = \min_{e\in C} u_f(e)$). We also decrease the circulation between edges in the "reverse" cycle, which might lead to some back-and-forth in which an edge that is part of two different negative cost cycles in opposite directions would change the direction of its circulation as we improved the circulation multiple times. But this would eventually end because, each time the circulation direction is reversed for that edge, the circulation is increased along all the other edges of the cycle in that direction, so eventually it will reach the capacity of one of them and the cycle will no longer be in the residual graph.

We can start with the empty circulation f_0, where $f_0(u,v) = 0$ for all $u, v \in V$. Then G_f is the same as G, but with edges of capacity 0 removed (if any). Then we repeat step (iii) until we have a minimum cost circulation.

Min-Circulation(G)

$f \leftarrow f_0$
while G_f has a negative cost cycle
  do $C \leftarrow$ some negative cost cycle in G_f
    $\delta \leftarrow \min_{e\in C} u_f(e)$
    for each $(u,v) \in C$
      do $f(u,v) \leftarrow f(u,v) + \delta$
        $f(v,u) \leftarrow f(v,u) - \delta$
return f