COMP 362 Assignment 5

Christopher Hundt 110220945

April 8, 2004

1. We call this problem SUBSET-COVER.

   We first consider that a certificate for this problem can be the desired subset S' (which is clearly smaller in size than the input). Using a naive method, we can simply look through S' for a member of each subset in C. This would take no more than $|S||C||S| = |S|^2|C|$ time, since $|S'| \subseteq S$ and each subset in C is no bigger than S. Thus we can verify a YES-instance in polynomial time. Thus SUBSET-COVER $\in$ NP.

   We now show that VERTEX-COVER $\leq$ SUBSET-COVER. Let (G, k) be an instance of VERTEX-COVER, where $G = (V, E)$. Then we define $R(G, k) = (S, C, K)$, where

   - $S = V$,
   - $C = \{\{u, v\} \mid (u, v) \in E\}$, and
   - $K = k$.

   Suppose $(G, k) \in$ VERTEX-COVER. Then there is some subset $V' \subseteq V$ of cardinality no more than k such that every edge in E has at least one endpoint in V'. Let $S' = V'$. S' has size no more than K, since $k = K$. Then, for each $\{u, v\} \in C$, we have $(u, v) \in E$, so either $u \in V'$ or $v \in V'$. Thus either $u \in S'$ or $v \in S'$. Therefore each member of C has at least one of its elements in S' and $|S'| \leq K$, so $R(G, k) = (S, C, K) \in$ SUBSET-COVER.

   Now suppose $R(G, k) = (S, C, K) \in$ SUBSET-COVER. Then let $V' = S'$. Then $|V'| = |S'| \leq K = k$. Also, for each edge $(u, v) \in E$, the set $\{u, v\} \in C$, so then $u \in S'$ or $v \in S'$. Thus either $u \in V'$ or $v \in V'$ for every edge. Thus V' is a subset of V of cardinality no more than k such that, for each edge in E, at least one endpoint is in V'. Thus $(G, k) \in$ VERTEX-COVER.

   Finally, we note that the construction of $R(G, k)$ involves nothing more than a direct copying of the input, with perhaps a slight change in format, so the time to compute it is obviously polynomial.

   Thus SUBSET-COVER is NP-complete.

2. We call this problem Isomorphic-Subgraph.

   First, note that we can use the subset V'' and the function f as a certificate. $|V''| \leq |V|$ and f can be encoded in maximum time proportional to $|V|$ (since f is surjective and $|V'| = |V''|$, so f is bijective), so the certificate is of size polynomially related to the input size. Then, we can naively check the V'' meets the given conditions by looping through the members of V'' and, for each one, looping through E to find related edges and then looping through f (which has no more than $|V''|$ transitions) and then E' and f again to make sure that these edges are matched. Thus the maximum time is proportional to $|V''||E||V''||E'||V''| = |V'|^3|E||E'|$, so it takes time polynomial with the input to verify a YES-instance. Thus Isomorphic-Subgraph $\in$ NP.

   We now show that Clique $\leq$ Isomorphic-Subgraph. Given an instance (G, k) of Clique, we define $R(G, k) = (G, H)$ where H is the fully connected graph with k vertices.

   Suppose $(G, k) \in$ Clique where $G = (V, E)$. Then there exists some subset $V'' \subset V$ of size k such that every vertex in V'' has an edge in E to every other vertex in V''. Thus (V'', E''), where $E'' = \{(u, v) \in E \mid u, v \in V''\}$, is a fully connected graph of size k and the isomorphism is obvious. Thus $R(G, k) = (G, H) \in$ Isomorphic-Subgraph.

   Now suppose $R(G, k) = (G, H) \in$ Isomorphic-Subgraph where $G = (V, E)$. Then there is an isomorphism between a subgraph of G and the fully connected graph of size k. Thus V contains a fully connected subset of size k, so $(G, k) \in$ Clique.

   Thus Isomorphic-Subgraph is NP-complete.

3. We call this problem Cycle-Cover.

   We use the subset V' as a certificate. We then take $V \setminus V'$, which we can do in polynomial time, and check with DFS (in polynomial time) whether it has any cycles. If $V \setminus V'$ has no cycle, then any previously existing cycle was removed by eliminating one of its vertices, so V' had a vertex in each cycle. If $V \setminus V'$ has a cycle C, then none of the vertices for C were in V', so V' did not cover all the cycles in G. Thus this test verifies a yes-instance. Thus Cycle-Cover $\in$ NP.

   We note that Vertex-Cover $\leq$ Cycle-Cover. For an instance (G, k) of Vertex-Cover we define $R(G, k) = (G', k)$, where $G' = (V, E')$ and $E' = \{(u, v) \mid (u, v) \in E\}$. That is, G' is G with each undirected edge replaced with a directed edge in each direction.

   Suppose $(G, k) \in$ Vertex-Cover where $G = (V, E)$ and $G' = (V, E')$ as defined above. Then there is a subset $V' \subset V$ with $|V'| \leq k$ such that every edge in E has at least one endpoint in V'. Then, for that subset V' in the graph G', every edge in E' has at least one endpoint in V'. Then, since directed cycles are made of edges, it is clear that at least one vertex from every cycle in G' is in V'. Thus $R(G, k) = (G', k) \in$ Cycle-Cover.

   Now suppose $R(G, k) = (G', k) \in$ Cycle-Cover where $G = (V, E)$ and $G' = (V, E')$ as defined above. Then there is a subset $V' \subseteq V$ with $|V'| = k$ such that V' contains at least one vertex from every directed cycle of G'. But we defined G' so that every edge has a symmetric back-edge with which it forms a cycle of length two. Such cycles involve only two vertices, and both edges in the cycle have both vertices as endpoints. Thus, if every such cycle has a vertex in V', then every edge in E' has an endpoint in V'. But since every directed edge in E' corresponds to an undirected edge in E with the same endpoints, every edge in E has an endpoint in V'. Thus V' is a subset of V of size k such that every edge in E has at least one endpoint in V', and $(G, k) \in$ Vertex-Cover.

   The time to compute $R(G, k)$ is polynomial in the input since the only modification is to iterate through the edges in E and add two edges in G' for each one.

   Thus Cycle-Cover is NP-complete.